Worcester County Mathematics League

All answers must be in simplest exact form

NO CALCULATOR ALLOWED

1. The difference between two numbers is 40. If seven is added to both of them, then the larger number becomes three times the smaller number. Find the value of the smaller of the original two numbers.

2. The average of a set of five numbers is 23. If two numbers are added to the set, the average of the new set becomes 29. What is the average of the two numbers that were added to the set?

3. A tank can be emptied by a drain in 4½ hours and filled by an inlet pipe in 3 hours. With the tank full at 11:00 am, the drain is opened and left open. At 2:00 pm the inlet pipe is turned on with the drain still open. At exactly what time will the tank be full again? Include am or pm in your answer.
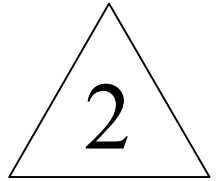
ANSWERS

(1 pt.)   1._____

(2 pts.)   2. _____

(3 pts.)   3._____

Worcester County Mathematics League

Freshman Meet 2 – January 4, 2012
Round 2: Number Theory

All answers must be in simplest exact form

NO CALCULATOR ALLOWED

Note: A subscript indicates the number's base.

$$\boxed{2}$$

1. How many of the first 1,000 positive integers are divisible by all of 2, 3, 5 and 7?

2. If A is replaced by a certain digit, then the sum written below is correctly worked out in base B:

$$
\begin{array}{cccc}
 & 7 & 6 & 3_B \\
+ & & A & 2_B \\
\hline
1 & 0 & 2 & 5_B
\end{array}
$$

   Compute the value of $A + B$.

3. Find the greatest common factor of 3,054 and 12,378.

ANSWERS

(1 pt.)   1._____

(2 pts.)   2._____

(3 pts.)   3._____

Southbridge, Worcester Academy, Leicester

Worcester County Mathematics League

Freshman Meet 2 – January 4, 2012
Round 3: Operations on Fractions, Decimals, Percents
and Percentage Word Problems

All answers must be in simplest exact form

NO CALCULATOR ALLOWED

$\langle 3 \rangle$

1. The product of the repeating decimals $0.\overline{3}$ and $0.\overline{6}$ is the repeating decimal $0.\overline{X}$. Find the value of X.

2. Sixty percent of the people in Imaginarytown are married. Of those people, 40% are under 30 years old and $33\frac{1}{3}\%$ are over 40 years old. What percentage of people in Imaginarytown are married and between the ages of 30 and 40 years old (inclusive)?

3. The complex fraction below can be simplified to a mixed number. If the mixed number is $A\frac{b}{c}$, where A is as large as possible and b and c are relatively prime, compute the sum $A + b + c$.

$$\frac{\left(\dfrac{3}{5} + \dfrac{5}{3}\right) \div \left(\dfrac{5}{3} - \dfrac{3}{5}\right)}{\dfrac{1}{\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4}}}$$

ANSWERS

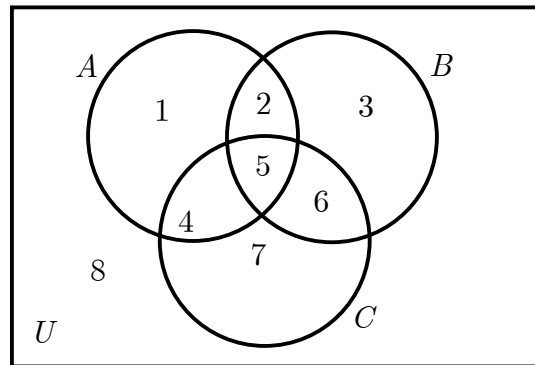(1 pt.)   1._____

(2 pts.)   2._____ %

(3 pts.)   3._____

All answers must be in simplest exact form

NO CALCULATOR ALLOWED

4

1. Let the set $Z = \{w, o, c, m, a, l\}$. How many four element subsets of set Z contain the element a?

2. The Venn diagram below shows sets A, B and C along with the universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$. If S' indicates the complement of set S, list the elements in the set given by $\left(\left[A' \cap (B \cap C)\right] \cup (A \cap B)\right) \cup A$.

3. Wachusett's indoor track team plans on ordering pizza after this weekend's meet. Thirteen of the team's members like sausage topping, 10 like pepperoni, 12 like extra cheese, 4 like both sausage and pepperoni, 5 like both pepperoni and extra cheese, 7 like both sausage and extra cheese, and 3 like all three toppings. Find the number of members on Wachusett's indoor track team.

ANSWERS

(1 pt.)   1._____ subsets

(2 pts.)   2. _____

(3 pts.)   3._____ members

Worcester County Mathematics League

Freshman Meet 2 – January 4, 2012
TEAM ROUND

All answers must *either* be in <u>simplest exact form</u> *or* as <u>decimals rounded</u> <u>correctly to at least three decimal places!</u>   (3 pts. each)

APPROVED CALCULATORS ALLOWED

1. The arithmetic mean of five numbers is 76, the median is 75, the mode is 81, and the range is 11. Find the value of the second smallest number.

2. I first drove 16 miles at 48 miles per hour (mph), then I drove 20 miles at 40mph, and finally I drove 24 miles at 36 mph. What was my average speed, in mph, for the entire trip?

3. How many liters of a 10% salt solution must be added to 6 liters of a 5% salt solution to yield a solution that is 8% salt?

4. If all of the 6-digit numbers formed by using the digits 1, 2, 3, 4, 5, and 6, without repetition, are listed from least to greatest, what number is in the 500th position?

5. My aunt's area code is a positive 3-digit number. If 7 is added to the number, the result is divisible by 7. If 8 is added to the number, the result is divisible by 8. If 9 is added to the number, the result is divisible by 9. What is my aunt's area code?

6. The point $(k, 0)$ is equidistant from the origin and the point $(2, 6)$. Compute the value of k.

7. On an eight-member math team, seven members like algebra, six members like number theory, and five members like geometry. What is the smallest possible number of members who like all three subjects?

8. The area of one side of a rectangular box is 126 square centimeters. The area of another side of the box is 153 square centimeters. If the area of the top of the box is 238 square centimeters, find the volume of the box (in cubic centimeters).

Algonquin, Hudson, Tantasqua, St. John's, Holy Name, Bromfield, AMSA Charter, QSC

Worcester County Mathematics League

Round 1

1. 13

2. 44

3. 8:00 pm   (or 8 pm)
      "pm" is necessary

Round 2

1. 4

2. 12

3. 6

Round 3

1. 2

2. 16

3. 127

Round 4

1. 10

2. 1, 2, 4, 5, 6   or   {1, 2, 4, 5, 6}
      the elements can be written in any order

3. 22

Team Round

1. 73

2. 40

3. 9

4. 516243

5. 504

6. 10

7. 2

8. 2142

Worcester County Mathematics League
Freshman Meet 2 – January 4, 2012
BRIEF SOLUTIONS

Round 1

1. Let $x =$ the smaller number, so that $x + 40$ is the larger number. We have $x + 47 = 3(x + 7)$
$\Rightarrow x + 47 = 3x + 21 \Rightarrow 2x = 26 \Rightarrow x = 13$

2. The sum of the first 5 numbers is $5 \cdot 23 = 115$. The sum of all seven numbers is $7 \cdot 29 = 203$. Therefore, the sum of the two numbers added to the original set is $203 - 115 = 88$. As a result, the average of the two numbers that were added is $\dfrac{88}{2} = 44$.

3. The tank drains in 4½ hours so $\dfrac{2}{9}$ of the tank drains in one hour. Also, the tank can be filled in 3 hours, so it fills at the rate of $\dfrac{1}{3}$ of the tank per hour. Let x be the number of hours after 2 pm. If "1" represents a full tank, we have $1 - \dfrac{2}{9}(x + 3) + \dfrac{1}{3}x = 1 \Rightarrow -\dfrac{2}{9}x - \dfrac{6}{9} + \dfrac{3}{9}x = 0 \Rightarrow \dfrac{1}{9}x = \dfrac{6}{9} \Rightarrow x = 6$. So, the tank is full again at 8:00 pm.

Round 2

1. Numbers divisible by 2, 3, 5, and 7 are multiples of the least common multiple of 2, 3, 5, and 7. The LCM of 2, 3, 5, and 7 is $2 \cdot 3 \cdot 5 \cdot 7 = 210$. Hence, the multiples of 210 less than 1,000 are 210, 420, 630, and 840 $\Rightarrow$ there are 4 such numbers.

2. First, note that $6 + A = 12_B$. Then, carrying the one gives $1 + 7 = 10_B \Rightarrow B = 8$. Hence,
$6 + A = 12_8 \Rightarrow 6 + A = 10 \Rightarrow A = 4$. Therefore, $A + B = 4 + 8 = 12$.

3. The prime factorization of 3,054 is $2 \times 3 \times 509$ (509 is prime by checking divisibility up through 19 since $22 < \sqrt{509} < 23$) and the prime factorization of 12,378 is $2 \times 3 \times 2063$ (2063 is prime by checking divisibility up through 43 since $44 < \sqrt{2063} < 45$). Therefore, the GCF of 3,054 and 12,378 is $2 \times 3 = 6$.

Round 3

1. $0.\overline{3} = \dfrac{1}{3}$ and $0.\overline{6} = \dfrac{2}{3}$ so that their product is $\dfrac{1}{3} \cdot \dfrac{2}{3} = \dfrac{2}{9} = 0.\overline{2}$.

2. Of the married people, $40\% = \dfrac{2}{5}$ are under 30 years old and $33\frac{1}{3}\% = \dfrac{1}{3}$ are over 40 years old, so that
$\dfrac{2}{5} + \dfrac{1}{3} = \dfrac{6 + 5}{15} = \dfrac{11}{15}$ are either under 30 or over 40. This implies that $\dfrac{4}{15}$ of the married people are between the ages of 30 and 40 (inclusive). As a result, the total fraction of married people between 30 and 40 is
$\dfrac{4}{15} \cdot \dfrac{3}{5} = \dfrac{4}{25} = 16\%$.

3. $\dfrac{3}{5}+\dfrac{5}{3}=\dfrac{34}{15}$, $\dfrac{5}{3}-\dfrac{3}{5}=\dfrac{16}{15}$, and $\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{4}=\dfrac{13}{12}$ so that the original expression is $\dfrac{\dfrac{34}{15}\cdot\dfrac{15}{16}}{\dfrac{12}{13}}$ or

$\dfrac{34}{15}\cdot\dfrac{15}{16}\cdot\dfrac{13}{12}=\dfrac{221}{96}=2\frac{29}{96}\Rightarrow A+b+c=2+29+96=127$.

Round 4

1. Z has 6 elements. The subsets to be counted contain "a" and three other elements chosen from the five

   remaining elements of Z. Therefore, there are $_5C_3=\dfrac{5!}{2!\cdot3!}=\dfrac{5\cdot4}{2}=10$ such subsets.

2. $A'=\{3,6,7,8\}$ and $B\cap C=\{5,6\}$ so that $A'\cap(B\cap C)=\{6\}$ Next, $A\cap B=\{2,5\}$ so that

   $A'\cap(B\cap C)\cup(A\cap B)=\{2,5,6\}$. Finally, $\left(A'\cap(B\cap C)\cup(A\cap B)\right)\cup A=\{1,2,4,5,6\}$.

3. With help from a Venn diagram (to the right), the number of members is
   $5+1+4+4+3+2+3=22$.

sausage   pepperoni   5   1   4   3   4   2   3   Extra Cheese
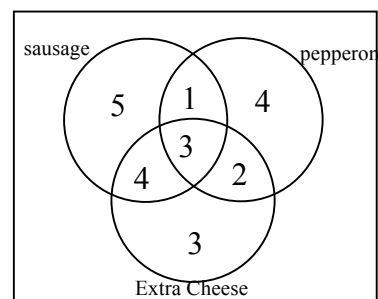
Team Round

1. If we call the numbers a, b, c, d, and e such that $a\le b\le c\le d\le e$, we know that
   $c=75$. Moreover, $d=e=81$ which leads to $a=70$. The sum of the numbers
   is $5\cdot76=380$. Therefore, $b=73$.

2. Driving 16 miles at 48 mph takes $\dfrac{16}{48}=\dfrac{1}{3}$ hour or 20 minutes. Driving 20 miles at 40 mph takes ½ and hour

   or 30 minutes. Driving 24 miles at 36 mph takes $\dfrac{24}{36}=\dfrac{2}{3}$ hour or 40 minutes. Therefore, I traveled

   $16+20+24=60$ miles in a total of $20+30+40=90$ minutes or 1.5 hours, and as a result, my average speed

   was $\dfrac{60}{1.5}=40$ mph.

3. Let $x=$ the # of liters of 10% solution that must be mixed in. We have $0.05\cdot6+0.1x=0.08(x+6)$
   $\Rightarrow30+10x=8(x+6)\Rightarrow2x=18\Rightarrow x=9$.

4. There are a total of $6!=720$ permutations of these digits. Of these, $5!=120$ begin with "1," 120 begin with
   "2," and so on. Thus, the last number that begins with "4" is the 480th number. $4!=24$ numbers begin
   with "51," so the 504th number is 516432. There are $3!=6$ numbers that begin with "512," 6 numbers that
   begin with "513," and 6 numbers that begin with "514." Thus the 498th number is 514632, the 499th is
   516234 and the 500th is 516243.

5. Let $A=$ my aunt's area code. If $A+7$ is divisible by 7, then A is also divisible by 7. Similarly, if $A+8$ and
   $A+9$ are divisible by 8 and 9, respectively, then A is also divisible by 8 and 9. Therefore, my aunt's area
   code is the least common multiple of 7, 8 and 9 or $7\cdot8\cdot9=504$.

6. Using the distance formula we know that $k=\sqrt{(k-2)^2+(6-0)^2}\Rightarrow k^2=k^2-4k+4+36\Rightarrow4k=40\Rightarrow k=10$.

7. Let $A=\{$members of who like algebra$\}$, $N=\{$members who like number theory$\}$, and $G=\{$members who
   like geometry$\}$, so that their complements are A', N' and G'. If $\left|S\right|=$ the number of elements in set S,

observe that $\left|A'\right| = 1$, $\left|N'\right| = 2$, and $\left|G'\right| = 3$. Therefore, there are at most $1 + 2 + 3 = 6$ students in the complement of $A \cap N \cap G$ which occurs when A', N', and G' are disjoint sets. Therefore, $A \cap N \cap G =$ {members who like all 3 subjects} has at least 2 members.

8. Let the dimensions of the box be x, y and z. We have $xz = 126$, $yz = 153$, and $xy = 238$. Multiplying the equations yields $(xyz)^2 = 126 \cdot 153 \cdot 238 = 4588164 \Rightarrow$ the volume of the solid is $xyz = \sqrt{4588164} = 2142$.